A CONSTRUCTION OF
HOMOLOGICALLY AREA MINIMIZING HYPERSURFACES
WITH HIGHER DIMENSIONAL SINGULAR SETS

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ABSTRACT. We show that a large variety of singular sets can occur for homo-
logically area minimizing codimension one surfaces in a Riemannian manifold.
In particular, as a result of Theorem A, if N is smooth, compact $n + 1$ di-
mensional manifold, $n \geq 7$, and if S is an embedded, orientable submanifold
of dimension n, then we construct metrics on N such that the homologically
area minimizing hypersurface M, homologous to S, has a singular set equal to
a prescribed number of spheres and tori of codimension less than $n - 7$. Near
each component Σ of the singular set, M is isometric to a product $C \times \Sigma$, where
C is any prescribed, strictly stable, strictly minimizing cone. In Theorem B,
other singular examples are constructed.

0. INTRODUCTION

In 1960, Federer and Flemming [6] developed the theory of rectifiable currents,
with which they were able to address some fundamental minimization problems in
geometry. The one we are interested in here, is the following. Given a smooth,
compact Riemannian manifold (N, g), of dimension $n + 1$, and given a non-zero,
integer homology class $\gamma \in H_k(N, \mathbf{Z})$, $1 \leq k \leq n$, does there exist a k-dimensional
surface $M \subset N$ (possibly with multiplicity), representing γ, which has the smallest
area among all such surfaces? This was answered affirmatively in [6] by allowing M
to be in a generalized class of surfaces called integral currents, which the authors
defined. A priori, an integral current is supported in a k-dimensional rectifiable
set, and thus it can have very general singularities. Indeed, if $k < n$ in the above
problem, there may be topological obstructions to the existence of a smooth sub-
manifold representing γ, although for the hypersurface case $(k = n)$, there are no
such obstructions. For codimension one, homologically area minimizing currents,
there is a partial regularity theory, [5], [11], and [4], which says that the singular
set of such currents has Hausdorff dimension at most $n - 7$, if $n \geq 7$ (consisting of
isolated points if $n = 7$), and is empty when $n < 7$. Such results also hold in the
context of area minimizing currents with a given boundary.

The first example of an area minimizing hypersurface with boundary, having a
singularity, was given by Bombieri, De Giorgi, and Giusti [2], where they showed
that the truncated cone over $S^p \times S^p$ in $\mathbf{R}^{2p+2}$ with $p \geq 3$, was area minimizing.
Further examples were given by Hardt and Simon [8], where they showed that
certain perturbations of truncated cones constructed earlier by Caffarelli, Hardt and

Simon [3], were area minimizing. In a previous paper [12], the author constructed the first examples of homologically area minimizing hypersurfaces in Riemannian manifolds, having singularities. There, it was shown that some smooth manifolds (in particular $S^n \times S^1$ with $n \geq 7$) admitted metrics such that the area minimizing currents in certain codimension one homology classes had two isolated singularities. Near the singularities, the minimizer looked like any prescribed strictly stable, strictly minimizing, regular hypercone (see [8] for a discussion of these cones). Here, using the same techniques, together with some new ideas, we show how to construct examples with higher dimensional singular sets. We can also, essentially prescribe the homology class. In particular we prove the following theorem.

Theorem A. *Let N be a smooth, compact Riemannian manifold of dimension $n+1 \geq 8$, and let S be a smooth, connected, oriented, embedded hypersurface which represents a nontrivial element of $H_n(N, \mathbf{Z})$. Let $\Sigma_1, \ldots, \Sigma_L$ be any collection of smooth manifolds, such that $\dim\Sigma_i = k_i \leq n - 7$, and each Σ_i is either a standard k_i-sphere (or a point if $k_i = 0$), or a k_i dimensional torus. Let $C_1, \ldots, C_L$ be a collection of strictly stable, strictly minimizing, regular hypercones, with $\dim C_i + k_i = n$. Then there is a metric g on N, such that the area minimizing current homologous to S, consists of a multiplicity one, connected hypersurface M, whose singular set is a union of submanifolds $\bigcup_i \Lambda_i$, where Λ_i is diffeomorphic to Σ_i. Near each Λ_i, M looks like a product $C_i \times \Sigma_i$.*

Actually, a somewhat stronger and more precise result is proven; the Σ_i can belong to a larger class of manifolds (see Theorem 1, of section 2). We also prove another result, which states that essentially any union of manifolds of appropriate codimension can occur as the singular set of some homologically minimizing current in some Riemannian manifold, although here we cannot prescribe the homology class, and the ambient manifold depends on the singular set one wishes to prescribe.

Theorem B. *Let $n \geq 7$, and let $\Sigma_1, \ldots, \Sigma_L$, be any collection of smooth, compact, oriented manifolds with $\dim\Sigma_i = k_i \leq n - 7$. Let $C_1, \ldots, C_L$ be any collection of strictly stable, strictly minimizing, regular hypercones with $\dim C_i + \dim\Sigma_i = n$. Then there is a smooth, compact, orientable Riemannian manifold (N, g) of dimension $n + 1$, which supports a homologically area minimizing current, which consists of a connected, multiplicity one hypersurface M, with singular set equal to a union $\bigcup_i \Lambda_i$, where each Λ_i is diffeomorphic to Σ_i. Near Λ_i, M looks like a product $C_i \times \Sigma_i$.*

See Theorem 2 in section 2, for a more precise result. The construction of N is fairly explicit, and there is a large class of such manifolds. The more difficult aspects of the proofs of the above theorems already appeared in [12]. In both theorems, the basic idea, is to first construct a hypersurface M, and a metric on N, so that M has the desired properties near it's singular set. The metric is then perturbed so that M is stationary. Then it is perturbed again so that M becomes stable (this was already done in [12]). It is then shown that if M is stable, and if the singular set has an appropriate product structure, then M is actually homologically minimizing in some neighborhood. The metric is then sufficiently enlarged on a complement of such a neighborhood, such that M becomes globally homologically minimizing.

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1. Some technical lemmas

In this section, (N, g) will denote a smooth, compact, Riemannian manifold of dimension $n + 1$, with $n \geq 7$. We will denote by M, a subset of N with certain regularity properties. On the complement of a closed singular set, $\operatorname{sing}(M)$ of Hausdorff dimension $\leq n - 7$, we will assume that $M/\operatorname{sing}(M)$ is a smooth, orientable, n-dimensional submanifold of N. Let $d(\cdot, \cdot)$ be the distance function in N (relative to g), and for a closed set $K \subset N$ let $d(x, K) = \inf_{y \in K} d(x, y)$. For $\sigma > 0$, and $\delta > 0$ define $\mathcal{N}(\sigma) = \{x \in N : d(x, \operatorname{sing}(M)) < \sigma\}$, and $\mathcal{U}(\delta) = \{x \in N : d(x, M) < \delta\}$. Recall that a regular, minimal hypercone C in $\mathbf{R}^{m+1}$, is a minimal surface of the form

$$C = \{r\theta : 0 \leq r < \infty, \ \theta \in \Theta\},$$

where Θ is a smooth $m - 1$-dimensional submanifold of the standard m-sphere. We will also use the notation $C(\sigma)$ ($\sigma > 0$), for the truncated cone $C \cap B^{m+1}(\sigma)$ where $B^{m+1}(\sigma)$ is the ball of radius σ centered at the origin in $\mathbf{R}^{m+1}$. We will frequently use the notions of stability, strict stability, and area minimizing, for hypersurfaces in Riemannian manifolds, as well as strict minimizing for cones in $\mathbf{R}^{m+1}$ (see section 2 of [12] and also [8]). In the first lemma, we assume that M is as above, and that near $\operatorname{sing}(M)$ M looks like a union of products of regular cones with complementary dimension manifolds. New metrics are then constructed so that M becomes homologically minimizing in a neighborhood.

Lemma 1. *Let M and (N, g_0) be as above, and suppose that there exists $\sigma > 0$ such that $\mathcal{N}(\sigma)$ is the disjoint union $\mathcal{N}(\sigma) = \bigcup_{i=1}^{k} \mathcal{N}_i(\sigma)$, and assume that there are isometries*

$$\Phi_i : \mathcal{N}_i(\sigma) \to B^{n_i+1}(\sigma) \times \Sigma_i,$$

where (Σ_i, h_i) is a compact Riemannian manifold of dimension k_i, $n_i + k_i = n$, $n_i \geq 7$, $k_i \geq 0$, and and the image of Φ_i is endowed with the product metric. Furthermore, assume that

$$\Phi_i(M \cap \mathcal{N}_i(\sigma)) = C_i(\sigma) \times \Sigma_i$$

where C_i is any strictly stable, strictly minimizing, regular hypercone in $\mathbf{R}^{n_i+1}$.

Then, there exists a metric g on N, with $g \equiv g_0$ on $\mathcal{N}(\sigma_1)$ for some $\sigma_1 < \sigma$, and a $\delta > 0$, such that M is the unique, homologically area minimizing current in $\mathcal{U}(\delta)$ relative to the metric g.

By homologically minimizing area in $\mathcal{U}(\delta)$, we mean that M, as considered as a multiplicity one, rectifiable current in $\mathcal{U}(\delta)$ minimizes mass among all rectifiable currents in $\mathcal{U}(\delta)$ homologous to M. Note of course, that the hypotheses on M imply that M is already area minimizing in a neighborhood of $\operatorname{sing}(M)$, with $\operatorname{sing}(M) = \bigcup_{i=1}^{k} \Phi_i^{-1}(0 \times \Sigma_i)$; the lemma allows us to change the metric away from the singular set so that M becomes locally homologically minimizing. Most of the technical difficulties in the construction have already been carried out in [12], and the needed modifications will be pointed out here. The proof follows from three propositions.

Proposition 1. *There is a metric g_1 on N, with that $g_1 \equiv g_0$ on $\mathcal{N}(\sigma/2)$, such that M is stationary in (N, g_1).*

Proof. We will conformally change g_0 in a neighborhood of M away from $\operatorname{sing}(M)$. That is, g_1 will have the form $g_1 = u_1 g_0$, where u_1 is a smooth, positive function

on N, with $u_1 \equiv 1$ outside a neighborhood of $M/\mathcal{N}(\sigma)$. Let H_0, and H_1 denote the mean curvature functions of M relative to the metrics g_0 and g_1 respectively, and let ν denote the outward unit normal to M (giving the positive orientation). Then, as computed in (2.9) of [12], we have

$$H_1 = u_1^{-1/2}(H_0 - \frac{n}{2u_1}\frac{\partial u_1}{\partial \nu}).$$

Thus M will be stationary relative to g_1 $(H_1 \equiv 0)$, provided that

$$(1.1) \qquad\qquad \frac{1}{u_1}\frac{\partial u_1}{\partial \nu} = \frac{2H_0}{n} \qquad \text{on } M.$$

There are many functions that satisfy (1.1) and the requirements of the proposition. For example, let (x, t) denote Fermi coordinates for N about $M/\mathcal{N}(\sigma/4)$. That is $x \in M/\mathcal{N}(\sigma/4)$ and $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and $t \to (x, t)$ is a unit speed geodesic perpendicular to M at each $x \in M$. Thus we can assume that $\frac{\partial}{\partial t} = \frac{\partial}{\partial \nu}$. Let u_1 be a smooth positive function on N such that

$$u_1(x, t) = \begin{cases} \exp\left(2/n \int_0^t H_0(x, s)ds\right) & \text{for } t \in (-\epsilon/2, \epsilon/2), \\ 1 & \text{for } |t| \geq 3\epsilon/4. \end{cases}$$

Since $H_0 \equiv 0$ on $M \cap \mathcal{N}(\sigma)$, u_1 is globally defined and clearly satisfies (1.1), and also $u_1 \equiv 1$ on $\mathcal{N}(\sigma/2)$ if ϵ is taken sufficiently small, depending on M and σ.

Proposition 2. *There exists a metric $g_2 = u_2 g_1$, where u_2 is a smooth positive function on N, with $u_2 \equiv 1$ on $\mathcal{N}(\sigma_1)$ for some $\sigma_1 < \sigma/2$, such that M is strictly stable in (N, g_2).*

Proof. With minor changes in notation, this is a special case of Lemma 1 in section 2 of [12]. Note that M satisfies the hypotheses of Lemma 1 of [12], being minimal in (N, g_1) (from Proposition 1), and also $M \cap \mathcal{N}(\sigma)$ is strictly stable in the sense (2.5) of [12] by the hypotheses on the structure of M near $\text{sing}(M)$. In Lemma 1 of [12], g_2 is constructed by creating negative Ricci curvature in N, in the direction normal to M, away from $\text{sing}(M)$, to make the Jacobi operator strictly positive.

The next proposition is analogous to Lemma 4, in section 3 of [12], and states that if M is stricly stable with a singular structure as above, then it is actually locally homologically area minimizing.

Proposition 3. *There exists a $\delta > 0$, such that M is homologically area minimizing in $\mathcal{U}(\delta)$ relative to the metric g_2.*

Proof. The proof is similar to the proof of Lemma 4 of [12], with some modifications. Since M is orientable, M divides $\mathcal{U}(\delta)$ into two disjoint open components, $\mathcal{U}^+(\delta)$ and $\mathcal{U}^-(\delta)$ for δ sufficiently small. Let (x, t) denote Fermi coordinates for N about M as in the proof of Proposition 1, valid for $|t| < \delta$, and for $x \in M/\mathcal{N}(\sigma_1/8)$, again for δ sufficiently small, depending on M, (N, g_2) and σ_1. Here also, t will be oriented so that $(x, t) \in \mathcal{U}^+(\delta)$ for $t > 0$. We will construct barrier hypersurfaces on both sides of M. Let $\rho = \rho(x) = \text{dist}(x, \text{sing}(M))$ for $x \in M$. For $|t| < \delta$, let

$$\Gamma_t = \{(x, t) : x \in M, \ \rho(x) \geq \sigma_1/8\},$$

That is, Γ_t is the graph of the constant function t over M. Letting h_t^Γ denote the mean curvature function of Γ_t, we have as in (3.1) of [12],

$$|h_t^\Gamma| \le c|t|,$$

for some constant c (depending on σ), as M is minimal. To construct the barriers near $\mathrm{sing}(M)$, define for each $i = 1, \ldots, k$, and $|t|$ sufficiently small (depending on σ_1)

$$\Lambda_t^i = \{(x,t) : \rho(x,0) = \sigma_1/4\}.$$

Let R_t^i be the area minimizing n-current in $\mathcal{N}_i(\sigma_1/4)$ with $\partial R_t^i = \Lambda_t^i$, $i = 1, \ldots, k$. By abuse of notation, we will also denote by R_t^i the support of R_t^i. Note that by our hypotheses on (N, g) and M, that $\Phi_i(R_t^i)$ is area minimizing in $B^{n_i+1}(\sigma_1/2) \times \Sigma_i$ with the product metric, with boundary the product

$$\partial\Phi_i(R_t^i) = \mathrm{graph}_{\partial C_i(\sigma_1/4)}\{t\} \times \Sigma_i.$$

Therefore $\Phi_i(R_t^i)$ is a product

$$\Phi_i(R_t^i) = S_t^i \times \Sigma_i,$$

where $S_t^i \subset B^{n_i+1}(\sigma_1/2)$ as an area minimizing n_i-current whose boundary is $\mathrm{graph}\{t\}$ over $\partial C_i(\sigma/4)$. By [8], $\mathrm{spt}S_t^i$ is a smooth hypersurface lying on one side of C_i. Furthermore, there is a $r_t > 0$ with $r_t \to 0$ as $|t| \to 0$, such that for $x \in B^{n_i+1}(\sigma_1/2)$ with $r_t \le |x| \le \sigma_1/4$, S_t^i is the graph over C_i of a smooth function w_i. Thus, after composition with Φ^{-1}, we see that R_t^i is the graph over M for $r_t \le \rho \le \sigma_1/4$, of a smooth function f_i (which is independent of the Σ_i variable). Denote by R_t the union $\bigcup_i R_t^i$. Of course the mean curvature of R_t is identically zero, and from above, R_t is the graph over M for $r_t \le \rho \le \sigma_1/2$ of some smooth function ψ_t. As in [12], we now glue together the R_t to the Γ_t along $\sigma_1/4 \le \rho \le \sigma_1$. Note that by the Harnack inequality (since $\psi_t > 0$ for $t > 0$) we have

$$c^{-1}|t| \le |\psi_t| \le c|t|, \qquad \text{for } \sigma_1/8 \le \rho \le \sigma_1/4,$$

for some constant c. By standard elliptic estimates [7], we then have

$$\max_{\sigma_1/7 \le \rho \le \sigma_1/5} |\nabla^j \psi_t| \le c|t|, \qquad k = 1, 2.$$

Let ζ be a smooth non-negative function on M such that

$$\zeta(x) = \begin{cases} 0 & \text{for } \rho \ge \sigma_1/5, \\ 1 & \text{for } \rho \le \sigma_1/7, \end{cases}$$

and let u_t be the smooth function defined on $M/\{\rho < \sigma_1/8\}$

$$u_t = \zeta\psi_t + (1 - \zeta)t.$$

It follows from the above estimates on ψ_t and the definition of u_t that

$$c^{-1}|t| \le |u_t| \le c|t| \qquad \text{for } \rho \ge \sigma_1/8,$$

$$\max_{\sigma_1/7 \le \rho \le \sigma_1/5} |\nabla^k u_t| \le c|t|, \qquad k = 1, 2.$$

Note that the graph of u_t (that is, in Fermi coordinates, the set $\{(x, u_t(x)) : x \in M, \ \rho(x) \ge \sigma_1/8\}$) contains the portion of Γ_t corresponding to $\rho(x) \ge \sigma_1/5$, and the graph of ψ_t for $\sigma_1/8 \le \rho(x) \le \sigma_1/7$. Let Ω_t be the smooth hypersurface obtained taking the union of the graph of u_t with the part of R_t corresponding to $\rho \le \sigma_1/8$. From here the proof proceeds as in the proof of Lemma 4 of [12], and

thus a concise sketch will be given here. By the comments above on the structure of R_t, it follows that Ω_t converges to M in Hausdorff distance as $t \to 0$. For $t > 0$, let V_t be the open component of $N/(\Omega_t \cup \Omega_{-t})$, that contains M. In proving the proposition, we may replace $\mathcal{U}(\delta)$ by V_t, since given $\delta > 0$ there is a $t > 0$ such that $V_t \subset \mathcal{U}(\delta)$, and conversely, given $t > 0$, there is a $\delta > 0$ such that $\mathcal{U}(\delta) \subset V_t$. Proving by contradiction, suppose that the proposition was false. Then there is a sequence $t_j \to 0$, and n-currents T_j in $\overline{V}_{t_j}$ such that $\mathbf{M}(T_j) < \mathbf{M}(M)$, with T_j homologous to M. Here $\mathbf{M}$ denotes the mass, and M is considered as a multiplicity one current. We may also assume by the compactness theorems of integral currents [6], that T_j minimizes mass among all n currents in $\overline{V}_{t_j}$ homologous to M. Since $\mathbf{M}(T_j)$ is bounded in j, there is a subsequence, still denoted by T_j and an n current T, homologous to M, such that $T_j \to T$ in the sense of currents. Clearly, $\mathrm{spt}(T) \subset \bigcap_j \overline{V}_{t_j} = M$, and so by the constancy theorem, we must have $T = M$. Note that T_j minimizes mass relative to the obstacle $\partial V_{t_j} = \Omega_{t_j} \cup \Omega_{-t_j}$, and so by [13], $\mathrm{spt}(T_j)$ is actually a $C^{1,1/2}$ manifold outside a singular set of dimension less than $n - 2$, and the mean curvature of T_j is bounded by the mean curvature of $\Omega_{t_j} \cup \Omega_{-t_j}$ which is bounded by $c|t_j| \to 0$, for some constant c independent of j. It then follows from Allard's regularity theorem ([1] or 24.2 of [9]) that there exists $\delta_j > 0$, with $\delta_j \to 0$, and $C^{1,1/2}$ functions v_j on $M/\mathcal{N}(\delta_j)$ such that for all j sufficiently large, $\mathrm{spt}(T_j)/\mathcal{N}(\delta_j) = \mathrm{graph}(v_j)$. The idea of the proof is to show that δ_j can be taken sufficiently small, so that v_j can be extended to all of M in such a way that the graph of v_j has smaller area than M, for large j (using of course $\mathbf{M}(T_j) < \mathbf{M}(M)$). This will then contradict strict stability of M (Proposition 2) which says more or less that M has smaller area than nearby graphs. Define s_j by

$$s_j = \max_{\rho \geq \sigma_1/8} |v_j| \leq ct_j,$$

for some constant c independent of j. Let $i \in \{1, \ldots, k\}$ be fixed, and denote by $\hat{T}_j$ the restriction of T_j to $\mathcal{N}_i(\sigma_1)$. Let W_j be the current

$$W_j = \Phi_{i\sharp}(\hat{T}_j).$$

Thus $\mathrm{spt} W_j \subset B^{n_i+1}(\sigma_1/4) \times \Sigma_i$, and lies between the barriers $S^i_{-s_j} \times \Sigma_i$ and $S^i_{s_j} \times \Sigma_i$. Since the barriers are area minimizing in $B^{n_i+1} \times \Sigma_i$, it follows from [13] that W_j is also area minimizing. But then by the maximum principle [10], $\mathrm{spt} W_j$ lies between $\lambda_j^\pm S^\pm \times \Sigma_i$, inside $B^{n_i+1}(\sigma_1/8) \times \Sigma_i$, where $S^\pm$ are the smooth area minimizing hypersurfaces in $\mathbf{R}^{n_i+1}$ that lie on either side of C_i described in the proposition of §3 of [12] (see also §2 of [8]), and $\lambda_j^\pm$ are appropriate scalings depending on s_j. The choice of $\lambda_j^\pm$ is made so that the minimum height of $S^\pm$ over C_i at $r = \sigma_1/8$ is s_j. Recall that $S^\pm \cap \mathbf{R}^{n_i+1}/\{r < R_0\}$ is the graph over $C_i/C_i(R_0)$ of a function $u^\pm$, for some $R_0 > 0$. Thus $\lambda_j^\pm S^\pm \cap \mathbf{R}^{n_i+1}/\{r < R_j^\pm\}$ is the graph of some function u_j, and some $R_j^\pm \to 0$. In fact, using the asymptotic behavior of $u^\pm$ from [8], it was shown in §3 of [12], that $\lambda_j^\pm$ and $R_j^\pm$ can be chosen so that

$$c^{-1} s_j^{\frac{1}{1-\gamma}} \leq \lambda_j^\pm \leq c s_j^{\frac{1}{1-\gamma}},$$

and

$$R_j = R_j^\pm = b s_j^{\frac{1}{1-\gamma}},$$

for some constants b and c, and where

$$\gamma = \frac{2-n_i}{2} + \sqrt{\frac{(n_i-2)^2}{4} + \mu_1}$$

with μ_1 a constant depending on C_i, with the quantity inside the radical positive (due to strict stability of C_i). Now, let $\epsilon > 0$, $\epsilon << 1$, to be fixed later, and let $\sigma_j = R_j^{1-\epsilon}$. Then we have (see (3.13) [12]),

$$|u_j(y)| \leq c\sigma_j^{\frac{1-\epsilon\gamma}{1-\epsilon}} \leq c\sigma_j^{1+\beta},$$

for some constant c, some $\beta > 0$ depending on ϵ, and for $\sigma_j \leq |y| \leq \sigma_1/4$, $y \in C_i$. We therefore have

$$\text{dist}(\text{spt}W_j \cap (B^{n_i+1}(\sigma_1/8)/B^{n_i+1}(\sigma_j) \times \Sigma_i), C_i(\sigma_1/8)/C_i(\sigma_j) \times \Sigma_i) \leq c\sigma_j^{1+\beta},$$

where dist is the Hausdorff distance. It now follows that $\text{spt}W_j \cap (B(\sigma_1/8)/B(\sigma_j) \times \Sigma_i)$ is a graph over $C_i(\sigma_1/8)/C_i(\sigma_j) \times \Sigma_i$ for large j. Arguing by contradiction, suppose this was not the case. Then there are points $p_j = (q_j, m_j) \in C_i \times \Sigma_i$, $j \to \infty$, with $\sigma_j \leq |q_j| \leq \sigma_1/8$, such that $\text{spt}W_j$ is not a graph over the point p_j. After taking a subsequence, we may assume that

$$|q_j|^{-1}q_j \to q \in C_i \qquad \text{and} \qquad m_j \to m \in \Sigma_i,$$

and we also may assume that $|q_j| \to 0$ as $j \to \infty$; otherwise the desired result follows immediately from the above inequality and Allards regularity theorem (see [1] or chapter 5 of [9]). But then, if we let A be a normal coordinate neighborhood of m in Σ_i, the above inequality implies that

$$|q_j|^{-1} (\text{spt}W_j \cap (B(\sigma_1/8)/B(|q_j|/2) \times A)) \to C_i/C_i(1/2) \times \mathbf{R}^{k_i},$$

as $j \to \infty$. But then, as above, Allard's theorem implies that for j sufficiently large, $\text{spt}W_j$ is graphical over $C_i \times \Sigma_i$ near p_j. Of course σ_j and γ depend on i; denote

$$\sigma_j = \sigma_j(i), \qquad \gamma = \gamma(i), \qquad \text{for } i = 1, \ldots, k, \;\; j \to \infty.$$

Thus, after composition with Φ_i, v_j is defined on $M \cap \mathcal{N}_i(\sigma_1)$ for $\rho \geq \sigma_j(i)$, and furthermore we have the estimate

$$|v_j(x)|\sigma_j(i)^{-1-\beta} + |\nabla v_j(x)|\sigma_j(i)^{-\beta} \leq c \qquad \text{for } x \in \mathcal{N}(\sigma_1), \; \rho \geq \sigma_j,$$

for a constant c independent of j. The first term on the left is bounded since $|v_j|$ is bounded by $|u_j^{\pm}|$ over this domain, while the bound on the second term follows from the bound on the first, and on standard elliptic estimates [7]. Now we extend v_j to a function $\tilde{v}_j$ defined on all of M (see 3.15) of [12] such that

$$\tilde{v}_j(x) = v_j(x) \qquad \text{for } x \in M/\bigcup_{i=1}^{k} \mathcal{N}(2\sigma_j(i)),$$

$$\tilde{v}_j(x) = 0 \qquad \text{for } x \in \bigcup_{i=1}^{k} \mathcal{N}(\sigma_j(i)),$$

$$|\tilde{v}_j(x)|\sigma_j(i)^{-1-\beta} + |\nabla \tilde{v}_j(x)|\sigma_j(i)^{-\beta} \leq c \qquad \text{for } x \in \bigcup_{i=1}^{k} \mathcal{N}(2\sigma_j(i))/\mathcal{N}(\sigma_j(i)),$$

for some constant c. Let $\tilde{T}_j$ be the multiplicity one current corresponding to the graph of $\tilde{v}_j$, oriented so that it is homologous to M. Now, $\text{spt}\tilde{T}_j = \text{spt}T_j$ outside of

$\bigcup_i \mathcal{N}_i(2\sigma_j(i))$, and the masses of each, inside $\mathcal{N}_i(2\sigma_j(i))$ are bounded by a constant times $\sigma_j(i)^{n_i}$, and therefore, the mass of $\tilde{T}_j$ inside $\mathcal{N}_i(2\sigma_j(i))$, is bounded by

$$c \sum_{i=1}^{k} \sigma_j(i)^{n_i} \le c \sum_{i=1}^{k} s_j^{\frac{1-\epsilon}{1-\gamma_i}n_i},$$

for some constant c. However, by the definition of γ_i one can fix $\epsilon > 0$ such that

$$\frac{1-\epsilon}{1-\gamma_i}n_i > 2, \qquad \text{for } i = 1, \ldots, k.$$

It then follows that there is a $\beta_1 > 0$, such that

(1.2) $$\mathbf{M}(\tilde{T}_j) - \mathbf{M}(T_j) \le cs_j^{2+\beta_1}.$$

The rest of the argument now proceeds exactly as in §3 of 12. Letting $A(v)$ denote the area of the graph of a C^1 function v on M, with sufficiently small C^1 norm, supported away from $\text{sing}(M)$, we have by Taylors theorem, and the first and second variation formulas (together with M stationary),

$$A(\tilde{v}_j) = A(0) + \int_M |\nabla \tilde{v}_j|^2 - (|B|^2 + R)\tilde{v}_j^2 \, dv + E(\tilde{v}_j),$$

where $|B|^2$ is the second fundamental form of M squared, R is the Ricci curvature of N in the normal direction of M, dv is the volume form of M, and E is the third order remainder. But strict stability of M (see §2 of [12]) implies that

$$\int_M |\nabla \tilde{v}_j|^2 - (|B|^2 + R)\tilde{v}_j^2 \, dv \ge \mu \int_M \tilde{v}_j^2 \rho^{-2} \, dv,$$

for a positive constant μ. A computation of $E(\tilde{v}_j)$ ((3.20)-(3.29) of [12]), together with Allards regularity theorem, standard Schauder estimates, and the above estimates on v_j, reveals that (see (3.19) [12])

$$E(\tilde{v}_j) \le \mu/10 \int_M \tilde{v}_j^2 \rho^{-2} \, dv,$$

for sufficiently large j. Therefore we have,

(1.3) $$\mathbf{M}(\tilde{T}_j) \ge \mathbf{M}(M) + \mu/4 \int_M \tilde{v}_j^2 \rho^{-2} \, dv,$$

again for $j \to \infty$. However, $\tilde{v}_j$ satisfies the estimate ((3.18) of [12])

$$\int_M \tilde{v}_j^2 \rho^{-2} \, dv \ge cs_j^2,$$

which follows from elliptic estimates for solutions of divergence form, quasilinear elliptic equations [7], after one notes that (weakly)

$$\mathcal{H}(v_j) = h_j \qquad \text{on } M/\mathcal{N}(\sigma_1/8)$$

where $\mathcal{H}$ is the mean curvature operator on M, and that the mean curvature h_j of v_j is bounded by a constant times s_j. Thus (1.3) becomes

$$\mathbf{M}(\tilde{T}_j) \ge \mathbf{M}(M) + cs_j^2,$$

for a positive constant c, and j large. Combining this with (1.2), and taking j large, this implies that

$$\mathbf{M}(M) < \mathbf{M}(T_j),$$

contradicting our assumption, which finishes the proof of Proposition 3, and thus of Lemma 1.

Lemma 2. *Let (N, g_0) be smooth, compact, $n+1$-dimensional Riemannian manifold, and suppose that M is a multiplicity one integral n-current which is uniquely homologically area minimizing in some neighborhood $\mathcal{U}$ of sptM, and that M is homologically nontrivial in N. Then there is a smooth Riemannian metric g on N, such that $g \equiv g_0$ on some smaller neighborhood $\mathcal{V} \subset \mathcal{U}$, of sptM, such that relative to g, M is homologically area minimizing in N.*

Such metrics were constructed in §4 of [12]. There, M was a specific hypersurface with two isolated singularities that had the hypotheses of the lemma, however the singular structure of M was never used in the proof, only the fact that M was uniquely homologically minimizing in a neighborhood $\mathcal{U}$. Then a smaller neighborhood $\mathcal{V}'$ of M was chosen with two smooth boundary components. It was then shown that if W was any smooth compact, connected manifold with ∂W diffeomorphic to $\partial \mathcal{V}'$, then W could be smoothly connected to $\mathcal{V}'$ along their boundaries, and a metric could be put on the resulting manifold (identical to the original metric on a smaller neighborhood $\mathcal{V}$ of M), in such a way that M was globally homologically area minimizing. Thus the same construction could be applied here where W is simply the complement in N of such a neighborhood $\mathcal{V}'$ of M with $\mathcal{V}' \subset \mathcal{U}$.

2. Constructing global homological minimizers

We first state and prove Theorem 1, which implies Theorem A of the introduction. For convenience, we will denote by $\mathcal{F}(n + 1)$, $n \geq 7$, the following class of smooth manifolds. A manifold Σ will be in $\mathcal{F}(n + 1)$ if and only if Σ is smooth, connected, compact, orientable, has dimension less than or equal to $n - 7$ and satisfies the the topological condition that there exists a smooth embedding $\Phi : \overline{B}^{m+1} \times \Sigma \to \mathbf{R}^{n+1}$, where $m + 1 + \dim\Sigma = n + 1$, and where $\overline{B}^{m+1}$ is the closure of the open unit ball in $\mathbf{R}^{m+1}$.

Remark. One can easily check that if Σ is any standard sphere or torus of dimension less than or equal to $n-7$, then $\Sigma \in \mathcal{F}(n+1)$, and thus Theorem 1 implies Theorem A. In fact, it is not hard to see, that if there exists an embedding of Σ into $\mathbf{R}^{n+1}$ with trivial normal bundle, then $\Sigma \in \mathcal{F}(n+1)$. In particular, if $\dim\Sigma = k \leq n-7$, and there is an embedding of Σ into $\mathbf{R}^{k+1}$, then $\Sigma \in \mathcal{F}(n+1)$ (assuming of course that Σ is smooth, connected and compact).

Theorem 1 allows us to construct homologically area minimizing hypersurfaces, with singular set any finite union of submanifolds belonging to $\mathcal{F}(n + 1)$.

Theorem 1. *Let N be a smooth, compact manifold of dimension $n+1 \geq 8$, and let S be a smooth, embedded, connected, orientable submanifold of N of dimension n, such that S is homologically non-trivial. Let $(\Sigma_1, h_1), \ldots, (\Sigma_L, h_L)$, $L \geq 1$, be any collection of Riemannian manifolds such that $\Sigma_i \in \mathcal{F}(n+1)$, and define $k_i = dim\Sigma_i$, $i = 1, \ldots, L$. Let $C_1, \ldots, C_L$ be any collection of strictly stable, strictly minimizing regular hypercones, $C_i \subset \mathbf{R}^{n_i+1}$, with $n_i = n - k_i$. Then there is a metric g on N, such that the area minimizing current homologous to S, is a multiplicity one,*

connected hypersurface M, with

$$sing(M) = \bigcup_{i=1}^{L} \Lambda_i,$$

where Λ_i is an embedded copy of Σ_i, $i = 1, \ldots, L$. Furthermore, near Λ_i, N is isometric to $B^{n_i+1}(\sigma) \times \Sigma_i$ with the product metric $\delta \times h_i$ (δ being the Euclidean metric) with M isometric to $C_i(\sigma) \times \Sigma_i$, for some σ sufficiently small.

Proof. The following proposition is useful in the construction of M. It says essentially, that we can "cap off" a truncated cone, and follows from some elementary facts from topology.

Proposition. *Let C be a regular minimal hypercone in $\mathbf{R}^{m+1}$, $m \geq 2$. Then there exists a compact hypersurface, without boundary, $\hat{C} \subset \mathbf{R}^{m+1}$, which is smoothly embedded except at the origin, and a $\sigma > 0$, such that $\hat{C} \cap B^{m+1}(\sigma) = C \cap B^{m+1}(\sigma)$. Furthermore, $\hat{C}$ is contained in the unit ball.*

Proof of Proposition. Consider the truncated cone $C(1) = B^{m+1}(1) \cap C$. The boundary, $\partial C(1)$ is a smoothly embedded $m-1$-dimensional submanifold of $\mathbf{R}^{m+1}$, and therefore (from basic results of topology) it bounds a smooth, compact n-dimensional manifold X. Then $Y = X \cup C(1)$ is a piecewise, C^1, compact embedded hypersurface, away from $\{0\}$. Now, outside of a sufficiently small ball $B^{m+1}(\epsilon)$, approximate Y by a smooth embedding, whose image we denote by $\hat{Y}$, such that $\hat{Y} \cap B^{m+1}(2\epsilon) = Y \cap B^{m+1}(2\epsilon)$. Finally, if we set $\hat{C}$ to be a sufficiently small scaling of $\hat{Y}$, then $\hat{C}$ satisfies the requirements of the proposition.

Now, since S is orientable, there is a neighborhood $\mathcal{E}$ of S, such that S divides $\mathcal{E}$ into two components, $\mathcal{E}^+$ and $\mathcal{E}^-$. Let $p_1, \ldots, p_L$ be in S, and let $\mathcal{B}_1, \ldots, \mathcal{B}_L$ be pairwise disjoint open sets of N diffeomorphic to $n+1$-balls, such that for each i, $p_i \in \mathcal{B}_i \subset \mathcal{E}$, and so that S divides $\mathcal{B}_i$ into two open components $\mathcal{B}^{\pm} \subset \mathcal{E}^{\pm}$.

For each $i = 1, \ldots, L$, we will do the following construction. Let $\hat{C}_i$ be a capped off C_i in B^{n_i+1} (the unit ball in $\mathbf{R}^{n_i+1}$ centered at the origin) as in the proposition. It follows from the hypotheses on Σ_i, that there is an embedding (into) $\Phi_i : B^{n_i+1} \times \Sigma_i \to \mathcal{B}_i^+$. Denote by $M_i = \Phi_i(\hat{C}_i \times \Sigma_i)$, and let D_i be a n-disc in M_i in the image of $(B^{n_i+1}(1)/B^{n_i+1}(1/2)) \times \Sigma_i$, and let D_i' be a n-disc in $S \cap \mathcal{B}_i$. Delete D_i and D_i', and smoothly connect S and M_i by a handle (a hypersurface in N diffeomorphic to an $n-1$-sphere times an interval) in $\mathcal{B}_i^+$, glueing the boundaries of the handle onto D_i and D_i'. Let M be the resulting hypersurface in N, that is the connected sum of S with M_i, $i = 1, \ldots, L$. The connected sum operation can be done so that M is embedded away from $sing(M)$. Note of course that then, M is orientable, and homologous to S (as a current) if given the right orientation. Let g_i be the pullback of the product metric on $\Phi_i(B^{n_i+1} \times \Sigma_i)$. That is, if δ_i denotes the Euclidean metric on $\mathbf{R}^{n_i+1}$, then set

$$g_i = (\Phi_i^{-1})^*(\delta_i \times h_i).$$

Using a standard partition of unity argument, we can then put a metric g_0 on N such that on each $\Phi_i(B^{n_i+1}(1/2) \times \Sigma_i)$, $g_0 = g_i$, $i = 1, \ldots, L$. But then, (N, g_0) and M satisfy the hypotheses of Lemma 1, and we can thus find a metric h on N, with $h \equiv g_0$ on a smaller neighborhood of $sing(M)$, such that M is homologically minimizing in a tubular neighborhood. But then, we can apply Lemma 2 to find a metric g on N which satisfies the requirements of Theorem 1.

The next theorem allows us to construct examples of homologically minimizing hypersurfaces, with singular sets of essentially arbitrary topological type (at least in the category of smooth submanifolds of appropriate codimension). However, unlike Theorem 1, we cannot prescribe the homology class or the ambient manifold.

Theorem 2. *Let $n \geq 7$, and let $(\Sigma_1, h_1), \ldots, (\Sigma_L, h_L)$ be smooth, compact, orientable Riemannian manifolds, with $\dim \Sigma_i = k_i \leq n - 7$. Let $X_1, \ldots, X_L$ be smooth, compact orientable manifolds with $\dim X_i = n_i + 1$ where $n_i + k_i = n$, and let $C_1, \ldots, C_L$ be strictly stable, strictly minimizing, regular hypercones with $C_i \subset \mathbf{R}^{n_i+1}$, $i = 1, \ldots, L$. Then we can construct an $n+1$-manifold N by forming a connected sum $(X_1 \times \Sigma_1) \# \cdots \# (X_L \times \Sigma_L)$, together with a handle attached, and a metric g on N, such that N supports a homologically area minimizing current M, which is a multiplicity one, connected hypersurface, with $\mathrm{sing}(M) = \bigcup_i \Lambda_i$ where, for each i, Λ_i is a diffeomorphic copy of Σ_i in N. Near each Λ_i, M looks (metrically) like the product $C_i \times \Sigma_i$.*

Proof. As in the proof of Theorem 1, let B^{n_i+1} denote the unit ball centered at the origin in $\mathbf{R}^{n_i+1}$, and let $\hat{C}_i$ be a capped off C_i in B^{n_i+1} (as in the proposition), $i = 1, \ldots, L$. For each i, let $F_i : B^{n_i+1} \to X_i$ be a smooth embedding, and let g_i be a metric on X_i such that

$$g_i \equiv (F_i^{-1})^*(\delta_i) \qquad \text{on } F_i(B^{n_i+1}(1/2)),$$

and equip $X_i \times \Sigma_i$ with the product metric $g_i \times h_i$. Let D_i be an $n+1$-disk in $X_i \times \Sigma_i$ contained in the complement of $F_i(B^{n_i+1}(1/2)) \times \Sigma_i$, $i = 1, \ldots, L$. Now form the connected sum

$$N' = \mathop{\#}_{i=1}^{L} (X_i \times \Sigma_i),$$

by deleting each D_i from $X_i \times \Sigma_i$, and smoothly glueing ∂D_i to ∂D_{i+1} (preserving orientation), $i \leq L - 1$. Put a metric g' on N' which coincides with $g_i \times h_i$ on each $(X_i \times \Sigma_i)/D_i$. Note that each $F_i(\hat{C}_i) \times \Sigma_i$ sits in N', and form a pairwise disjoint collection of hypersurfaces. Now, for $i = 1, \ldots, L-1$, smoothly connect $F_i(\hat{C}_i) \times \Sigma_i$ to $F_{i+1}(\hat{C}_{i+1}) \times \Sigma_{i+1}$ by a handle in N' (as in the proof of Theorem 1), attaching them at the boundaries of n-disks, well away from the singular set. Let M denote the resulting hypersurface. Of course, M is homologically trivial in N', however, it is easy to see that it divides N' into two open components N'_+ and N'_-. Form a smooth, oriented $n+1$-manifold, N by attaching a handle from N'_+ to N'_-. That is, delete $n+1$-disks from N'_+ and N'_- and let N be the union of these, together with $Y = n - \text{sphere} \times (0, 1)$, where the boundary components of Y are identified with the boundaries of the deleted disks by suitable diffeomorphisms. Finally, put a metric G on N which coincides with g' on N'. Now M is homologous to a nontrivial $n - \text{sphere} \times \{\text{point}\} \subset Y \subset N$. Also, M satisfies the hypotheses of Lemma 1, so we can find a metric h on N, which is identical to G in a neighborhood of $\mathrm{sing}(M)$, and so that M is homologically minimizing in a tubular neighborhood. Then, apply Lemma 2, to get a metric g on N, which satisfies the requirements of Theorem 2.

References

1. W.K. Allard, *On the first variation of a varifold*, Ann. Math. **95** (1972), 417–491. MR **46**:6136
2. E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268. MR **40**:3445

3. L. Caffarelli, R. Hardt, L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. **48** (1984), 1–18. MR **86h:**53007

4. E. De Giorgi, *Frontiere orientate di misura minima*, Sem. Mat. Scuola Norm. Sup. Pisa (1961), 1–56. MR **31:**3897

5. H. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970), 767–771. MR **41:**5601

6. H. Federer, W. Fleming, *Normal and integral currents*, Ann. Math. **72** (1960), 458–520. MR **23:**A588

7. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (2nd ed.), Springer-Verlag, 1983. MR **86c:**35035

8. R. Hardt, L. Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129. MR **87f:**49052

9. L. Simon, *Lectures on Geometric Measure Theory,* Proceedings for the Centre for Mathematical Analysis, Australian National University, Canberra, 1983. MR **87a:**49001

10. L. Simon, *A strict maximum principle for area minimizing hypersurfaces*, J. Differential Geom. **26** (1987), 327–335. MR **88h:**49069

11. J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. Math. **88** (1968), 62–105. MR **38:**1617

12. N. Smale, *Singular homologically area minimizing surfaces of codimension one in Riemannian manifolds*, Invent. Math. **135** (1999), 145–183. CMP 99:06

13. B. White, *Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on 3-manifolds*, J. Differential Geom. **33** (1991), 413–443. MR **92e:**58048

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